in the problem of maximizing the plate stiffness.

In conclusion, we note that the Weierstrass-Erdmann condition for the stiffness minimization problem will be satisfied on discontinuities of $\theta^*(\mathbf{x})$ while the Weierstrass condition will not be satified at points $\mathbf{x}$ in which $\theta_- \leqslant \theta^*(\mathbf{x}) \leqslant \theta_+$.

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ON THE STATE OF STRESS AND STRAIN NEAR CONE APICES*

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The asymptotic form of the state of stress and strain near the apices of inclusions or cavities having the form of a pointed cone is investigated. An arbitrary simple closed contour in a plane bounding a set g_ε of a small parameter ε is the directrix of the conical surface. The principal term of the asymptotic form $\varepsilon^2\Lambda_2 + O(\varepsilon^3)$ of the stress singularity index is calculated and examples are considered. The problem of the axisymmetric strain of an elastic half-space with a thin conical recess is investigated.

1. A pointed conical inclusion and recess. Let k_ε denote a thin cone $\{\mathbf{x} \in \mathbf{R}^3: x_3 > 0, \varepsilon^{-1}x_3^{-1}\mathbf{x}' \in g, \mathbf{x}' = (x_1, x_2)\}$, where ε is a small positive parameter, and g is a domain in the plane bounded by a simple smooth contour ∂g. We will consider the cones k_ε and $K_\varepsilon = \mathbf{R}^3 \setminus \bar{k}_\varepsilon$ filled with elastic isotropic materials with Lamé constants λ°, μ° and λ, μ, respectively, and the material contact is ideal (without peeling and slippage). It is known that the behaviour of the state of stress and strain near a conical point O is governed by the eigennumbers and vectors of a certain eigenvalue problem in the domain cut out of the cone by a unit sphere S. We introduce spherical coordinates (ρ, θ, φ), where $\rho = |\mathbf{x}|, \theta \in [0, \pi]$ is the latitude, $\varphi \in [0, 2\pi)$ is the longitude, and $\rho^{-2}Q(\theta, \varphi, \rho\partial/\partial\rho, \partial/\partial\theta, \partial/\partial\varphi)$ will denote the matrix operator of the Lamé system. We write the stress vector normal to the surface ∂K_ε in an analogous form $\rho^{-1}P(\theta, \varphi, \rho\partial/\partial\rho, \partial/\partial\theta, \partial/\partial\varphi)\mathbf{u}$. Here $\mathbf{u}$ is the displacement vector. (To abbreviate the notation, the arguments θ, φ and $\partial/\partial\theta, \partial/\partial\varphi$ will not be indicated everywhere later.). Let g_ε° be the set cut out by the cone k_ε on the sphere S. The problem with the complex spectrum parameter $\Lambda(\varepsilon)$ has the form

$$Q(\Lambda(\varepsilon))\mathbf{v} = 0 \text{ on } S \setminus g_\varepsilon^\circ \tag{1.1}$$

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$$Q^\circ(\Lambda(\varepsilon))\,\mathbf{v}^\circ = 0 \text{ on } g_\varepsilon^\circ \tag{1.2}$$

$$\mathbf{v} = \mathbf{v}^\circ,\ P(\Lambda(\varepsilon))\,\mathbf{v} = P^\circ(\Lambda(\varepsilon))\,\mathbf{v}^\circ \text{ on } \partial g_\varepsilon^\circ \tag{1.3}$$

All the quantities referring to the inclusion k_ε are given the symbol $^\circ$.

The special vector-functions $\rho^{\Lambda(\varepsilon)}\mathbf{V}(\varepsilon, \theta, \varphi, \ln\rho)$, $\rho^{\Lambda(\varepsilon)}\mathbf{V}^\circ(\varepsilon, \theta, \varphi, \ln\rho)$, occur in the asymptotic expansion of the displacements $\mathbf{u}, \mathbf{u}_0$ near the conical point, where $\mathbf{V}$ and $\mathbf{V}^\circ$ are polynomials in the variable $\ln\rho$ whose coefficients are the eigenvectors and associated vectors of problem (1.1)-(1.3) corresponding to the eigenvalue $\Lambda(\varepsilon)$. We emphasize that the exact answers (the transcendental equations for the indices) are known only for the axisymmetric problem in the case of a circular conical inclusion or cavity /1-7/; the transcendental equation mentioned requires numerical solution; tables of values of the singularity index can be found in /1, 4-7/.

We will use the algorithm in /8/ to determine the asymptotic behaviour of several first positive eigenvalues of problem (1.1)-(1.3) as $\varepsilon \to 0$.

As $\varepsilon \to 0$ the domain g_ε° vanishes in the limit and problem (1.1)-(1.3) is transformed into a system of equations on the sphere S without a hole

$$Q(\Lambda_0)\,\boldsymbol{\Phi} = 0 \tag{1.4}$$

((1.2) and (1.3) are not taken into account here). It is easy to enumerate all the solutions of the spectral problem (1.4): the eigennumbers Λ_0 are integers, and the vectors $\boldsymbol{\Phi}$ are traces on S of homogeneous vector polynomials $\mathbf{V}^{(m,j)}$ $(m = 0, 1, 2, \ldots, j = 1, 2, \ldots, 3(2m+1))$ of degree m that satisfy the Lamé system, or traces of the fields $\mathbf{V}^{(1,j)}(\partial/\partial\mathbf{x})\,T(\mathbf{x})$, where T is the Somigliani tensor. Since solutions with a finite elastic energy are considered, only special solutions in which $\Lambda(\varepsilon) > -1/2$ can occur in the asymptotic form. Consequently, we study perturbations of just the first two eigenvalues $\Lambda_0 = 0$ and $\Lambda_0 = 1$ of system (1.4). Since the vectors $\mathbf{V}^{(0,j)}$ $(j = 1, 2, 3)$ correspond to rigid translational displacements, they satisfy problem (1.1)-(1.3), $\Lambda(\varepsilon) = 0$. The vector polynomials $\mathbf{V}^{(1,j)}$ of first degree have the form

$$\mathbf{V}^{(1,1)}(\mathbf{x}) = (x_1, 0, 0),\ \mathbf{V}^{(1,2)}(\mathbf{x}) = (0, x_2, 0),\ \mathbf{V}^{(1,3)}(\mathbf{x}) = (0, 0, x_3), \tag{1.5}$$
$$\mathbf{V}^{(1,4)}(\mathbf{x}) = 2^{-1/2}(x_2, x_1, 0),\ \mathbf{V}^{(1,5)}(\mathbf{x}) = 2^{-1/2}(0, x_3, x_2),\ \mathbf{V}^{(1,6)}(\mathbf{x}) = 2^{-1/2}(x_3, 0, x_1)$$

$$\mathbf{V}^{(1,7)}(\mathbf{x}) = 2^{-1/2}(x_2, -x_1, 0),\ \mathbf{V}^{(1,8)}(\mathbf{x}) = 2^{-1/2}(0, x_3, -x_2) \tag{1.6}$$
$$\mathbf{V}^{(1,9)}(\mathbf{x}) = 2^{-1/2}(-x_3, 0, x_1)$$

Eqs.(1.1) and (1.2) are true for the traces $\boldsymbol{\Phi}^{(1,j)}$ of the rotations (1.6), which means that even in this case the eigenvalue $\Lambda_0 = 1$ is not perturbed. The traces $\boldsymbol{\Phi}^{(1,j)}$ $(j = 1, 2, \ldots, 6)$ of the fields (1.5) on the sphere S leave residuals in the conjugate conditions (1.3). We note that the stresses $\sigma_{jk}^{(i)} = \sigma_{jk}(\mathbf{V}^{(1,i)})$ are evaluated by the following formulas:

$$\sigma_{ii}^{(i)} = 2\mu + \lambda,\ \sigma_{ii}^{(j)} = \lambda,\quad i \neq j,\ i, j = 1, 2, 3; \tag{1.7}$$
$$\sigma_{12}^{(4)} = \sigma_{23}^{(5)} = \sigma_{13}^{(6)} = \sqrt{2}\mu$$

The components equal to zero are not indicated; analogous expressions hold in the inclusions.

Thus, we take the number $\Lambda_0 = 1$ and linear combinations with coefficients c_j and c_j° (to be determined)

$$\boldsymbol{\Phi}(\theta, \varphi) = \sum_{j=1}^{9} c_j \boldsymbol{\Phi}^{(1,j)}(\theta, \varphi),\quad \boldsymbol{\Phi}^\circ(\theta, \varphi) = \sum_{j=1}^{9} c_j^\circ \boldsymbol{\Phi}^{(1,j)}(\theta, \varphi) \tag{1.8}$$

as the fundamental approximation to the solution of problem (1.1)-(1.3).

2. ***Boundary layer near*** g_ε°. We introduce the coordinate $\eta = (\eta_1, \eta_2) = x_3^{-1}\mathbf{x}'$ and the "expanded" coordinate $\xi = \varepsilon^{-1}\eta$ in the neighbourhood of the north pole $\mathbf{N} = (0, 0, 1)$ on the sphere S. Since the vector of the unit normal $\mathbf{n}$ on ∂k_ε equals $(1 + \varepsilon^2(\xi\cdot\mathbf{v})^2)^{-1/2}(\nu_1, \nu_2, -\varepsilon\xi\cdot\mathbf{v})$, where $\mathbf{v}(\xi)$ is the vector of the internal unit normal to ∂g in a plane, then the equalities

$$L\left(\frac{\partial}{\partial\mathbf{x}}\right)(\rho^{1+O(\varepsilon^2)}\boldsymbol{\Psi}(\xi))\Big|_{|\mathbf{x}|=1} = \varepsilon^{-2}L_0\left(\frac{\partial}{\partial\xi}\right)\boldsymbol{\Psi}(\xi) + \varepsilon^{-1}L_1\left(\xi, \frac{\partial}{\partial\xi}\right)\boldsymbol{\Psi}(\xi) + O(1) \tag{2.1}$$

$$B\left(\frac{\partial}{\partial\mathbf{x}}\right)(\rho^{1+O(\varepsilon^2)}\boldsymbol{\Psi}(\xi))\Big|_{|\mathbf{x}|=1} = \varepsilon^{-1}B_0\left(\frac{\partial}{\partial\xi}\right)\boldsymbol{\Psi}(\xi) + B_1\left(\xi, \frac{\partial}{\partial\xi}\right)\boldsymbol{\Psi}(\xi) + O(\varepsilon) \tag{2.2}$$

$$L_0^{11}(\zeta_1, \zeta_2) = (\lambda + 2\mu)\zeta_1^2 + \mu\zeta_2^2, \; L_0^{12}(\zeta_1, \zeta_2) = L_0^{21}(\zeta_1, \zeta_2) = (\lambda + \mu)\zeta_1\zeta_2, \; L_0^{22}(\zeta_1, \zeta_2) = (\lambda + 2\mu)\zeta_2^2 + \mu\zeta_1^2$$
$$L_0^{33}(\zeta_1, \zeta_2) = \mu(\zeta_1^2 + \zeta_2^2), \; L_1^{13}(\xi_1, \xi_2; \zeta_1, \zeta_2) = L_1^{31}(\xi_1, \xi_2; \zeta_1, \zeta_2) = -(\lambda + \mu)(\xi_1\zeta_1^2 + \xi_2\zeta_1\zeta_2)$$
$$L_1^{23}(\xi_1, \xi_2; \zeta_1, \zeta_2) = L_1^{32}(\xi_1, \xi_2; \zeta_1, \zeta_2) = -(\lambda + \mu)(\xi_1\zeta_1\zeta_2 + \xi_2\zeta_2^2), \; B_0^{11}(\xi; \zeta_1, \zeta_2) = (\lambda + 2\mu)\nu_1\zeta_1 + \mu\nu_2\zeta_2, \; B_0^{22}(\xi; \zeta_1, \zeta_2) = \mu\nu_1\zeta_1 + (\lambda + 2\mu)\nu_2\zeta_2, \; B_0^{12}(\xi; \zeta_1, \zeta_2) = \lambda\nu_1\zeta_2 + \mu\nu_2\zeta_1, \; B_0^{21}(\xi; \zeta_1, \zeta_2) = \mu\nu_1\zeta_2 + \lambda\nu_2\zeta_1, \; B_0^{33}(\xi; \zeta_1, \zeta_2) = \mu(\nu_1\zeta_1 + \nu_2\zeta_2)$$
$$B_1^{j3}(\xi_1, \xi_2; \zeta_1, \zeta_2) = -\lambda\nu_j(\xi_1\zeta_1 + \xi_2\zeta_2 - 1) - \mu\xi\cdot\nu\zeta_j, \; B_1^{3j}(\xi_1, \xi_2; \zeta_1, \zeta_2) = -\mu\nu_j(\xi_1\zeta_1 + \xi_2\zeta_2 - 1) - \lambda\xi\cdot\nu\zeta_j, \; j = 1, 2; \; \nu = (\nu_1(\xi), \nu_2(\xi)) \tag{2.3}$$

hold.

We take the vector $\varepsilon\mathbf{w}^{(1)}(\xi)$, $\varepsilon\mathbf{w}^{\circ(1)}(\xi)$ as the principal term of the boundary layer. We will find the problem that they satisfy. The domain $g_\varepsilon^0 \subset S$ in the coordinates ξ coincides with the domain $g \in \mathbf{R}^2$. Consequently, the system of equations for $\mathbf{w}^{(1)}$ and $\mathbf{w}^{\circ(1)}$ in $\mathbf{R}^2 \setminus \bar{g}$ and g are determined, respectively, by using relationships (2.1). In order to derive the conjugate condition on ∂g, the relationship (2.2) and the residuals left by the quantities (1.8) in the second of the conditions (1.3) must be taken into account. These are calculated using (1.7). We findally obtain the problem

$$L_0\left(\frac{\partial}{\partial\xi}\right)\mathbf{w}^{(1)}(\xi) = 0, \quad \xi \in \mathbf{R}^2\setminus\bar{g}, \quad L_0^\circ\left(\frac{\partial}{\partial\xi}\right)\mathbf{w}^{\circ(1)}(\xi) = 0, \quad \xi \in g \tag{2.4}$$

$$B_0\left(\xi, \frac{\partial}{\partial\xi}\right)\mathbf{w}^{(1)}(\xi) - B_0^\circ\left(\xi, \frac{\partial}{\partial\xi}\right)\mathbf{w}^{\circ(1)}(\xi) = -\sum_{j=1}^{6} c_j\Psi^{(j)}(\xi) \tag{2.5}$$
$$\mathbf{w}^{(1)}(\xi) = \mathbf{w}^{\circ(1)}(\xi), \quad \xi \in \partial g$$

$$\Psi^{(1)}(\xi) = ((\lambda + 2\mu - \lambda^\circ - 2\mu^\circ)\nu_1, (\lambda - \lambda^\circ)\nu_2, 0)$$
$$\Psi^{(2)}(\xi) = ((\lambda - \lambda^\circ)\nu_1, (\lambda + 2\mu - \lambda^\circ - 2\mu^\circ)\nu_2, 0)$$
$$\Psi^{(3)}(\xi) = (\lambda - \lambda^\circ)(\nu_1, \nu_2, 0), \; \Psi^{(4)}(\xi) = 2^{1/2}(\mu - \mu^\circ)(\nu_2, \nu_1, 0)$$
$$\Psi^{(5)}(\xi) = 2^{1/2}(\mu - \mu^\circ)(0, 0, \nu_2), \; \Psi^{(6)}(\xi) = 2^{1/2}(\mu - \mu^\circ)(0, 0, \nu_1) \tag{2.6}$$

According to (2.3), the boundary-value problem (2.4) and (2.5) decomposes into two: a plane problem of elasticity theory (the first line) and a problem of antiplane shear (the third line). Since the mean quantities (2.6) in ∂g equal zero, a solution $\mathbf{w}^{(1)}$ of problem (2.4) and (2.5) exists that vanishes at infinity.

The following asymptotic formulas hold /9/

$$\mathbf{w}^{(1)}(\xi) = \Upsilon^{(1)}(\xi) + O(|\xi|^{-2}) = \sum_{j=1}^{6} c_j \sum_{i=1}^{5} \alpha_i^{(j)}\mathbf{W}^{(i)}(\partial/\partial\xi)\Gamma(\xi) + O(|\xi|^{-2}), \tag{2.7}$$
$$|\xi| \to \infty; \quad \mathbf{W}^{(1)}(\xi) = (\xi_1, 0, 0), \quad \mathbf{W}^{(2)}(\xi) = (0, \xi_2, 0),$$
$$\mathbf{W}^{(3)}(\xi) = 2^{-1/2}(\xi_2, \xi_1, 0), \quad \mathbf{W}^{(4)}(\xi) = (0, 0, \xi_1), \quad \mathbf{W}^{(5)}(\xi) = (0, 0, \xi_2),$$
$$\Gamma(\xi) = \|\gamma_{ij}(\xi)\|_{i,j=1}^{3}$$
$$\gamma_{ij}(\xi) = [4\pi\mu(\lambda + 2\mu)]^{-1}(-\delta_{ij}(\lambda + 3\mu)\ln|\xi| + (\lambda + \mu)\xi_i\xi_j|\xi|^{-2}), \; \gamma_{3j}(\xi) = \gamma_{j3}(\xi) = 0, \; i, j = 1, 2$$
$$\gamma_{33}(\xi) = -(2\pi\mu)^{-1}\ln|\xi|$$

The coefficients $\alpha_i^{(j)}$ are expressed in terms of the elastic polarization matrix elements $m = \|m_{ik}\|_{i,k=1}^{5}$ comprised of factors m_{ik} for $\mathbf{W}^{(i)}(\partial/\partial\xi)\Gamma(\xi)$ in the asymptotic representation of the form (2.7) for the special solutions $\mathbf{Z}^{(i)}$ of problem (2.4) and (2.5) with the right sides

$$((\lambda^\circ + 2\mu^\circ - \lambda - 2\mu)\nu_1, (\lambda^\circ - \lambda)\nu_2, 0), ((\lambda - \lambda^\circ)\nu_1, (\lambda^\circ + 2\mu^\circ - \lambda - 2\mu)\nu_2, 0), 2^{1/2}(\mu^\circ - \mu)(\nu_2, \nu_1, 0)$$
$$(\mu^\circ - \mu)(0, 0, \nu_1), (\mu^\circ - \mu)(0, 0, \nu_2) \tag{2.8}$$

We note that the polarization matrix is negative (positive) definite for sufficiently soft (hard) inclusions of non-zero volume. The above-mentioned connection between $\alpha_k^{(1)}$ and m_{jk} is given by the formulas

$$\alpha_k^{(j)} = m_{jk},\ j = 1,\ 2;\ \alpha_k^{(3)} = (\lambda - \lambda^\circ)\,[2\,(\mu - \mu^\circ + \lambda - \lambda^\circ)]^{-1}\,(m_{1k} + m_{2k}),\ \alpha_k^{(4)} = m_{3k},\ \alpha_k^{(j)} = 0,\ j = 5,\ 6,\ k = 1,\ 2,\ 3$$
$$\alpha_4^{(j)} = \alpha_5^{(j)} = 0,\ j = 1,\ 2,\ 3,\ 4;\ \alpha_k^{(k+1)} = 2^{1/2} m_{45},\ k = 4,\ 5$$
$$\alpha_4^{(6)} = 2^{1/2} m_{44},\ \alpha_5^{(5)} = 2^{1/2} m_{55}$$

By virtue of (2.7) the components of the vector $\mathbf{w}^{(1)}$ decrease as $O(|\xi|^{-1})$ and this means the boundary layer $\varepsilon\chi(\theta)\mathbf{w}^1(\varepsilon^1\eta)$ leaves the residual $O(\varepsilon^2)$ in Eqs.(1.1), $\Lambda(\varepsilon) = 1$. (Here χ is a truncating function, $\chi(\theta) = 1$ for $\theta \in [0, \pi/6]$ and $\chi(\theta) = 0$ for $\theta \in [\pi/3, \pi]$; it is introduced because the boundary layer is given only in the upper hemisphere). Therefore, the asymptotic form of the solution of problem (1.1)-(1.3) should be sought in the form

$$\Lambda(\varepsilon) \sim 1 + \varepsilon^2\Lambda_2,\ \mathbf{v}(\varepsilon, \theta, \varphi) \sim \mathbf{\Phi}(\theta, \varphi) + \varepsilon\chi(\theta)\,\mathbf{w}^{(1)}(\varepsilon^{-1}\eta) + \varepsilon^2\mathbf{\Phi}^{(2)}(\theta, \varphi) + \varepsilon^2\chi(\theta)\,\mathbf{w}^2(\varepsilon^{-1}\eta) \tag{2.9}$$

We will first determine the second term of the boundary-layer type solution. Taking account of (2.1), (2.2) and (1.7), we obtain that the vector $\mathbf{w}^{(2)}$ is a solution of the problem

$$L_0\mathbf{w}^{(2)} + L_1\mathbf{w}^{(1)} = 0 \text{ in } \mathbf{R}^2 \setminus g \tag{2.10}$$
$$L_0^\circ\mathbf{w}^{\circ(2)} + L_1^\circ\mathbf{w}^{\circ(1)} = 0 \text{ in } g \tag{2.11}$$
$$\mathbf{w}^{(2)} = \mathbf{w}^{\circ(2)},\quad B_0\mathbf{w}^{(2)} - B_0^\circ\mathbf{w}^{\circ(2)} = B_1^\circ\mathbf{w}^{(1)} - B_1\mathbf{w}^{(1)} + \sum_{j=1}^{6} c_j\mathbf{\Psi}^{(1,j)} \text{ on } \partial g \tag{2.12}$$

$$\mathbf{\Psi}^{(1,1)}(\xi) = \mathbf{\Psi}^{(1,2)}(\xi) = (0, 0, (\lambda - \lambda^\circ)\,\xi\cdot\mathbf{v}) \tag{2.13}$$
$$\mathbf{\Psi}^{(1,3)}(\xi) = (0, 0, (\lambda + 2\mu - \lambda^\circ - 2\mu^\circ)\,\xi\cdot\mathbf{v}),\quad \mathbf{\Psi}^{(1,4)}(\xi) = 0$$
$$\mathbf{\Psi}^{(1,5)}(\xi) = 2^{1/2}(0, (\mu - \mu^\circ)\,\xi\cdot\mathbf{v}, 0)$$
$$\mathbf{\Psi}^{(1,6)}(\xi) = 2^{1/2}((\mu - \mu^\circ)\,\xi\cdot\mathbf{v}, 0, 0)$$

Let us study the behaviour of the field $\mathbf{w}^{(2)}$ at infinity.

Proposition 1. Every solution $\mathbf{w}^{(2)}$ of (2.10) allowing the estimate $O(|\xi|^\delta)$ for $\delta \in (0, 1)$, has the asymptotic form

$$\mathbf{w}^{(2)}(\xi) = \Upsilon^{(2)}(\xi) + O(|\xi|^{-1}) = \mathbf{a}\Gamma(\xi) + \mathbf{b} + \Xi(\varphi) + O(|\xi|^{-1}) \tag{2.14}$$
$$\Xi(\xi) = (\pi\mu)^{-1}(\Xi_1^\circ,\ \Xi_2^\circ,\ \Xi_3^\circ)$$
$$\Xi_j^\circ(\xi) = \varkappa^{-1}\sum_{k=5}^{6} c_k \sum_{i=1}^{2} \alpha_{3+i}^{(k)}\xi_i\xi_j\,|\xi|^{-2},\quad j = 1, 2$$
$$\Xi_3^\circ(\xi) = (\varkappa + 1)^{-1}\sum_{j=1}^{4} c_j\,(\alpha_1^{(j)}\xi_1^2\,|\xi|^{-2} + \alpha_2^{(j)}\xi_2^2\,|\xi|^{-2} + 2^{1/2}\alpha_3^{(j)}\xi_1\xi_2\,|\xi|^{-2}),$$
$$\varkappa = (\lambda + 3\mu)(\lambda + \mu)^{-1} \tag{2.15}$$

Proof. By virtue of (2.7) $\Upsilon^{(1)}$ is a homogeneous vector function of degree -1. Since

$$\xi\cdot\nabla_\xi\,\partial/\partial\xi_j = \partial/\partial\xi_j\,|\xi|\,\partial/\partial|\xi| - \partial/\partial\xi_j$$

then according to (2.3)

$$L_1(\xi;\ \partial/\partial\xi)\,\Upsilon^{(1)} = 2(\lambda + \mu)(\Upsilon_{3,1}^{(1)},\ \Upsilon_{3,2}^{(1)},\ \Upsilon_{1,1}^{(1)} + \Upsilon_{2,2}^{(1)}) + O(|\xi|^{-3}) = |\xi|^{-2}\mathbf{\Theta}(\varphi) + O(|\xi|^{-3})$$

Here and later the subscript k after a comma denotes differentiation with respect to ξ_k.

Seeking the particular solution of the equation $L_0\Xi = -|\xi|^{-2}\mathbf{\Theta}$, we arrive at the equalities (2.15). It remains to note that $L_0(\mathbf{a}\Gamma + \mathbf{b}) = 0$ for $\xi \neq 0$, and the basis for (2.14) follows from the results in /9, 10/.

The solution of probelm (2.10)-(2.12) is determined to the accuracy of a constant vector, meaning the column $\mathbf{b}$ in (2.14) is arbitrary. Furthermore, it is convenient to consider that

$$\mathbf{b} = -(\ln\varepsilon)\,(2\pi\mu)^{-1}\,(\varkappa(\varkappa + 1)^{-1}a_1,\ \varkappa(\varkappa + 1)^{-1}a_2,\ a_3)$$

For such a selection of $\mathbf{b}$ the quantity $\Upsilon^{(2)}$, written in the coordinates $\eta = \varepsilon\xi$, is independent of the parameter ε. In order to evaluate the column **a**, we use the method described in /10/.

Proposition 2. The equalities

$$a_k = \sum_{j=1}^{6} c_j \beta_k^{(j)}, \quad k = 1, 2, 3; \quad \beta_1^{(p)} = 4\mu(\lambda + 3\mu)^{-1}\alpha_4^{(p)} \tag{2.16}$$
$$\beta_2^{(p)} = 4\mu(\lambda + 3\mu)^{-1}\alpha_5^{(p)}, \quad p = 5, \ 6; \ \beta_3^{(j)} = (A + B)(\alpha_1^{(j)} + \alpha_2^{(j)}) + 2\delta_{j3}[(\lambda - \lambda^\circ) A - (\lambda + 2\mu - \lambda^\circ - 2\mu^\circ)] \times \operatorname{mes}_2 g, \ j = 1, 2, 3, 4; \ A = (\lambda - \lambda^\circ)(\lambda + \mu - \lambda^\circ - \mu^\circ)^{-1}, \ B = -(\lambda + \mu)(\lambda + 2\mu)^{-1}$$

are valid.

Proof. We multiply system (2.10) and (2.11) scalarly by the unit vectors $\mathbf{e}^{(i)}$, we integrate by parts in a circle D_R^2 of radius R and then we pass to the limit as $R \to \infty$. We have

$$\int\limits_{D_R^2 \setminus g} \mathbf{e}^{(i)} \cdot (L_0 \mathbf{w}^{(2)} + L_1 \mathbf{w}^{(1)})\, d\xi + \int\limits_g \mathbf{e}^{(i)} \cdot (L_0^\circ \mathbf{w}^{\circ(2)} + L_1^\circ \mathbf{w}^{\circ(1)})\, d\xi = \tag{2.17}$$
$$\int\limits_{\partial D_R^2} \mathbf{e}^{(i)} \cdot (\mathbf{B}_0 \mathbf{w}^{(2)} + \mathbf{B}_1 \mathbf{w}^{(1)})\, dl + \int\limits_{\partial g} \mathbf{e}^{(i)} \cdot \sum_{j=1}^{6} c_j \boldsymbol{\Psi}^{(1, j)} dl + 2I_i$$
$$I_j = \mu R^{-1} \int\limits_{\partial D_R^2} \xi_j w_3^{(1)} dl + (\mu - \mu^\circ) \int\limits_{\partial g} w_3^{(1)} \nu_j dl, \quad j = 1, 2$$
$$I_3 = \sum_{j=1}^{2} \Big\{ \lambda R^{-1} \int\limits_{\partial D_R^2} \xi_j w_j^{(1)} dl + (\lambda - \lambda^\circ) \int\limits_{\partial g} w_j^{(1)} \nu_j dl \Big\}$$

Here $\mathbf{B}_0$ and $\mathbf{B}_1$ are operators given by (2.3) with the normal vector $\boldsymbol{\nu}$ replaced by the vector $(\cos\varphi, \sin\varphi)$. In order to evaluate the first two integrals on the right-hand side of (2.17), we note that the first of them equals

$$\int\limits_{\partial D_R^2} \mathbf{e}^{(i)} \cdot \mathbf{B}_0 \mathbf{a} \Gamma dl + \int\limits_{\partial D_R^2} \mathbf{e}^{(i)} \cdot (\mathbf{B}_0 \boldsymbol{\Xi} + \mathbf{B}_1 \boldsymbol{\Upsilon}^{(1)})\, dl + o(1) \tag{2.18}$$

where

$$\int\limits_{\partial D_R^2} \mathbf{e}^{(i)} \cdot \mathbf{B}_0 \mathbf{a} \Gamma dl = -\int\limits_{D_R^2} \mathbf{e}^{(i)} \cdot \mathbf{a}\delta(\xi)\, d\xi = -a_i, \quad \int\limits_{\partial g} \xi_j \nu_k dl = -\delta_{jk} \operatorname{mes}_2 g \tag{2.19}$$

The second integral on the right-hand side of (2.18) is found by direct calculations by using (2.15), (2.7), and (2.3). When considering the integrals I_j, the components of the normal must be expressed in terms of the vector $(B_0 - B_0^\circ)\xi_j \mathbf{e}^{(3)}$, $(B_0 - B_0^\circ)\xi_j \mathbf{e}^{(j)}$ $(j = 1, 2)$ and then the Betti formula must be used, as well as the asymptotic expansion of the vector $\mathbf{w}^{(1)}$ at infinity, and a transformation of the type (2.19). We consequently arrive at the relationships (2.16).

3. ***Definition of*** Λ_2. We will now evaluate the quantities $\boldsymbol{\Phi}^{(2)}$ and Λ_2 from the asymptotic form (2.9). Apart from the smallest terms, the operator $Q(1 + \varepsilon^2 \Lambda_2)$ is identical with the sum $Q(1) + \varepsilon^2 \Lambda_2 Q'(1)$, where the prime denotes a derivative with respect to t of the abstract function $t \to Q(t)$. Moreover, it follows from representation (2.1) that the relationship

$$Q(1) = L_0(\partial/\partial\eta) + L_1(\eta, \partial/\partial\eta) + L_2(\eta, \partial/\partial\eta) \tag{3.1}$$

is valid near **N**.

Here L_2 is a matrix differential operator in which the coefficients for derivatives or order k have the order $|\eta|^k$. Taking into account the residual $O(\varepsilon^2)$ that appears in system (1.1) because of the presence of a boundary layer, we conclude that the vector $\boldsymbol{\Phi}^{(2)}$ and the number Λ_2 satisfy the system

$$Q(1)\boldsymbol{\Phi}^{(2)} = -\Lambda_2 Q'(1)\boldsymbol{\Phi} - \mathbf{F} \text{ on } S, \ \mathbf{F} = L_2 \chi \boldsymbol{\Upsilon}^{(1)} + (Q(1) - L_0)\chi \boldsymbol{\Upsilon}^{(2)} + [L_p + L_1, \chi]\boldsymbol{\Upsilon}^{(1)} + [L_0, \chi]\boldsymbol{\Upsilon}^{(2)}, \quad [A, B] = AB - BA \tag{3.2}$$

Let us study the vector Eq.(3.2)

Proposition 3. The system $Q(1)\mathbf{V} = \mathbf{F}_*$ on S is solvable if and only if the equalities

$$\int_S \mathbf{F}_* \cdot \mathbf{Y}^{(1,j)}\, ds = 0, \quad j = 1, 2, \ldots, 9 \tag{3.3}$$

are valid, where $\mathbf{Y}^{(1,j)}$ are traces of the fields $\mathbf{V}^{(1,j)}(\partial/\partial\mathbf{x})\,T(\mathbf{x})$ on the sphere S. The solution $\mathbf{V}$ is determined apart from an arbitrary constant column $\mathbf{c}$.

Proposition 4. The following equalities hold:

$$\int_S \mathbf{Y}^{(1,k)} \cdot Q'(1)\,\mathbf{\Phi}^{(1,j)}\, ds = -\delta_{jk}, \quad j, k = 1, 2, \ldots, 9 \tag{3.4}$$

$$\int_S \mathbf{Y}^{(1,j)} \cdot \mathbf{F}\, ds = \sum_{k=1}^{9} M_{jk} c_k, \quad j = 1, 2, \ldots, 9 \tag{3.5}$$

$$\begin{gathered} M_{1j} = q\left[-(2-\varkappa)\alpha_1^{(j)} - \alpha_2^{(j)} + \beta_3^{(j)} + 2(\varkappa+1)^{-1}(\alpha_1^{(j)} + \alpha_2^{(j)})\right] \\ M_{2j} = q\left[-\alpha_1^{(j)} - (2-\varkappa)\alpha_2^{(j)} + \beta_3^{(j)} + 2(\varkappa+1)^{-1}(\alpha_1^{(j)} + \alpha_2^{(j)})\right] \\ M_{3j} = -q(\varkappa+1)\,\beta_3^{(j)},\ M_{4j} = -q(1-\varkappa)\,\alpha_3^{(j)},\ j = 1, 2, 3, 4 \\ M_{5p} = 2^{-1/2}q\left[(3+\varkappa)\,\alpha_5^{(p)} + (1-\varkappa)\,\beta_2^{(p)}\right],\ M_{6p} = 2^{-1/2}q\,[(3 + \\ \varkappa)\,\alpha_4^{(p)} + (1-\varkappa)\,\beta_1^{(p)}],\ M_{8p} = -2^{-1/2}q\,(\varkappa+1)\,[(2+\lambda\mu^{-1}) \times \\ \alpha_5^{(p)} + \beta_2^{(p)}],\ M_{9p} = 2^{-1/2}q\,(\varkappa+1)\,[(2+\lambda\mu^{-1})\,\alpha_4^{(p)} + \\ \beta_1^{(p)}],\ p = 5, 6;\ q = (\lambda+\mu)\,[8\pi\mu\,(\lambda+2\mu)]^{-1} \end{gathered} \tag{3.6}$$

Proof. Since the Lamé system operator is formally selfadjoint, then $Q^*(\Lambda) = Q(-1-\overline{\Lambda})$. Consequently, the first assertion results from the statements about homogeneous solutions of the Lamé system in Sect.1.

We verify (3.4). Let ζ be a function from $C_0^\infty[0,1)$ that equals one near zero, and let D_d^3 be a sphere of radius d with centre at O. According to the definition of the Somigliani tensor, we have

$$\begin{gathered} \int_{D_1^3} \mathbf{V}^{(1,k)}\left(\frac{\partial}{\partial\mathbf{x}}\right) T(\mathbf{x}) \cdot L\left(\frac{\partial}{\partial\mathbf{x}}\right)(\zeta(\rho)\,\mathbf{V}^{(1,j)}(\mathbf{x}))\, d\mathbf{x} = \\ \int_{D_1^3} \zeta(\rho)\,\mathbf{V}^{(1,j)}(\mathbf{x}) \cdot L\left(\frac{\partial}{\partial\mathbf{x}}\right)\mathbf{V}^{(1,k)}\left(\frac{\partial}{\partial\mathbf{x}}\right) T(\mathbf{x})\, d\mathbf{x} = \mathbf{V}^{(1,k)}(\partial/\partial\mathbf{x})\,\mathbf{V}^{(1,j)}(0) = \delta_{jk} \end{gathered} \tag{3.7}$$

On the other hand, since

$$Q\left(\rho\frac{\partial}{\partial\rho}\right)(\rho\mathbf{\Phi}^{(1,j)}) = \rho Q\left(1 + \rho\frac{\partial}{\partial\rho}\right)\mathbf{\Phi}^{(1,j)},\quad Q(\Lambda+1) = Q(1) + \Lambda Q'(1) + {}^1/_2\Lambda^2 Q''(1)$$

the chain of equalities is true that together with (3.7) yield (3.4) and

$$\begin{gathered} \int_{D_1^3} \mathbf{V}^{(1,k)}\left(\frac{\partial}{\partial\mathbf{x}}\right) T(\mathbf{x}) \cdot L\left(\frac{\partial}{\partial\mathbf{x}}\right)(\zeta(\rho)\,\mathbf{V}^{(1,j)}(\mathbf{x}))\, d\mathbf{x} = \lim_{d\to 0}\int_d^1\int_S \rho^{-2}\mathbf{Y}^{(1,k)}(\theta,\varphi)\,Q \times \\ Q\left(\rho\frac{\partial}{\partial\rho}\right)(\zeta(\rho)\,\rho\mathbf{\Phi}^{(1,j)}(\theta,\varphi))\, d\rho\, ds = \lim_{d\to 0}\int_d^1\int_S \left(\frac{\partial\zeta}{\partial\rho}\mathbf{Y}^{(1,k)}(\theta,\varphi)\,Q'(1)\,\mathbf{\Phi}^{(1,j)}(\theta,\varphi) - \right. \\ \left. \frac{1}{2}\frac{\partial\zeta}{\partial\rho}\mathbf{Y}^{(1,k)}(\theta,\varphi)\,Q''(1)\,\mathbf{\Phi}^{(1,j)}(\theta,\varphi) + \frac{1}{2}\frac{\partial\zeta}{\partial\rho}\mathbf{Y}^{(1,k)}(\theta,\varphi)\,Q''(1)\,\mathbf{\Phi}^{(1,j)}(\theta,\varphi)\right) d\rho\, ds = \\ -\lim_{d\to 0}\zeta(d)\int_S \mathbf{Y}^{(1,k)}(\theta,\varphi)\,Q'(1)\,\mathbf{\Phi}^{(1,j)}(\theta,\varphi)\, ds = -\int_S \mathbf{Y}^{(1,k)}(\theta,\varphi) \cdot Q'(1)\,\mathbf{\Phi}^{(1,j)}(\theta,\varphi)\, ds \end{gathered}$$

It remains to note that the equalities (3.5) are a result of relationships resulting from (3.1) and Proposition 3:

$$\begin{gathered} Q(1)(\chi(\theta)(\mathbf{\Upsilon}^{(1)}(\eta) + \mathbf{\Upsilon}^{(2)}(\eta))) = \mathbf{F}(\eta) - \chi(\theta)\sum_{j=1}^{6} c_j \times \\ \left(\sum_{i=1}^{5}\alpha_i^{(j)}\mathbf{W}^{(i)}\left(\frac{\partial}{\partial\eta}\right) + \sum_{i=1}^{3}(\beta_i^{(j)} + d_i^{(j)})\,\mathbf{e}^{(i)}\right)\delta(\eta) \\ d_i^{(p)} = -(\lambda+\mu)\,\mu^{-1}\alpha_{3+i}^{(p)},\ i = 1, 2,\ p = 5, 6;\ d_3^{(j)} = -(\lambda+\mu)(\lambda+2\mu)^{-1}(\alpha_1^{(j)} + \alpha_2^{(j)}), \\ j = 1, 2, 3, 4 \\ \int_S \mathbf{\Upsilon}^{(1,k)} \cdot Q(1)(\chi\mathbf{\Upsilon}^{(1)} + \chi\mathbf{\Upsilon}^{(2)})\, ds = 0 \end{gathered}$$

It follows from (3.4) and (3.5) that the conditions (3.3) for the vector Eq.(3.2) with right-hand side $\mathbf{F}_* = -\mathbf{F} - \Lambda_2 Q'(1)\mathbf{\Phi}$ to be solvable take the form of a system of linear algebraic equations with a spectral parameter, i.e., Λ_2 is an eigenvalue of the matrix M with elements (3.6) while the vector $\mathbf{c}$ of the coefficients of linear combinations (1.8) is the corresponding eigencolumn

$$M\mathbf{c} = \Lambda_2 \mathbf{c} \tag{3.8}$$

The matrix M has a block configuration. The eigenvectors $\mathbf{e}^{(7)}$, $\mathbf{e}^{(8)}$, $\mathbf{e}^{(9)}$, the unit vectors in $\mathbf{R}^9$, and the triple eigennumber $\Lambda_2 = 0$ correspond to the rotations $\mathbf{V}^{(1,7)}$, $\mathbf{V}^{(1,8)}$, $\mathbf{V}^{(1,9)}$ (see (1.6) and Sect.1). The 4x4 and 2x2 blocks of the matrix M generate two more groups of eigenvalues $\Lambda_2^{(j)}$ $(j = 1, 2, 3, 4)$ and $\Lambda_2^{(k)}$ $(k = 5, 6)$.

1°. *A thin crack of angular planform.* Let the cone K_ε be formed by removal of the set $\{x\colon x_2 = 0,\ x_3 \geqslant 0,\ |x_1| \leqslant \varepsilon x_3\}$ from the space $\mathbf{R}^3$. The corresponding set $\bar{g}_\varepsilon^0$ on the unit sphere S is the arc of a major circle of length $2\,\mathrm{arctg}\,\varepsilon$. (Note the in substance the requirement of smoothness of the contour ∂g was never used.) Two polarization matrix for the crack consists of the two blocks

$$-\frac{\pi(\lambda+2\mu)}{2\mu(\lambda+\mu)}\begin{Vmatrix} \lambda^2 & (\lambda+2\mu)\lambda \\ (\lambda+2\mu)\lambda & (\lambda+2\mu)^2 \end{Vmatrix},\ \mathrm{diag}\left(-\pi\mu\frac{\lambda+2\mu}{\lambda+\mu};\ 0;\ -\pi\mu\right)$$

Substituting the expressions for its elements into (3.6), we find that

$$\Lambda_2^{(5)} = -\frac{2\lambda^2+9\mu\lambda+5\mu^2}{4(\lambda+2\mu)(\lambda+3\mu)},\ \Lambda_2^{(6)} = 0$$

Moreover, the block of dimensions 4x4 mentioned earlier and its eigennumbers have the form

$$\frac{\mu^{-2}}{16}\begin{Vmatrix} \lambda t_1 & \lambda t_2 & -\lambda t_3 & 0 \\ (\lambda+2\mu)t_1 & (\lambda+2\mu)t_2 & -(\lambda+2\mu)t_3 & 0 \\ \lambda t_1 & \lambda t_2 & -\lambda t_3 & 0 \\ 0 & 0 & 0 & -4\mu^3(\lambda+\mu)^{-1} \end{Vmatrix}$$

$$t_1 = 2\mu + \lambda(1-\varkappa),\ t_2 = (2-\varkappa)(\lambda+2\mu) - \lambda,\ t_3 = 4(\lambda+\mu) - 2\lambda(\varkappa+1)$$
$$\Lambda_2^{(1)} = \Lambda_2^{(2)} = 0,\quad \Lambda_2^{(3)} = -{}^1/_4,\quad \Lambda_2^{(4)} = -\mu\,[4(\lambda+\mu)]^{-1}$$

We emphasize that the stresses in problems concerning the tension at infinity of a space with a narrow crack by the forces $\sigma_{33}^\infty, \sigma_{11}^\infty$ or σ_{13}^∞ are constant and therefore have no singularities. Finally, $\Lambda_2^{(3)} < 0$ and $\Lambda_2^{(4)} \in (-{}^1/_4,\ 0)$, $\Lambda_2^{(5)} \in (-{}^1/_2, -{}^5/_{24})$.

2°. Let k_ε be a circular cone $\{x\colon x_3 > 0,\ |x'| < \varepsilon x_3\}$. Then g is a unit circle and the corresponding polarization matrix is comprised of blocks

$$-\frac{\pi\lambda(+2\mu)}{\mu}\begin{Vmatrix} \lambda+\mu\varkappa & \lambda+\mu(2-\varkappa) & 0 \\ \lambda+\mu(2-\varkappa) & \lambda+\mu\varkappa & 0 \\ 0 & 0 & 2\mu(\varkappa-1) \end{Vmatrix},\ -2\pi\mu\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

The four eigenvalues of the matrix M are evaluated from the formulas

$$\Lambda_2^{(3)} = \Lambda_2^{(4)} = -\mu(\lambda+\mu)^{-1} \in (-1,\ 0),\ \Lambda_2^{(5)} = \Lambda_2^{(6)} = -(2\lambda^2 + 9\mu\lambda + 5\mu^2)\ \times$$
$$[2(\lambda+2\mu)(\lambda+3\mu)]^{-1} \in (-1;\ -{}^5/_{12})$$

They correspond to non-axisymmetric solutions. The axisymmetric components possess the singularities $\varepsilon^2\Lambda_2^{(i)} + O(\varepsilon^3)$, $i = 1, 2$, where

$$\Lambda_2^{(1)} = 0,\quad \Lambda_2^{(2)} = -\frac{5\lambda^2+9\mu\lambda+2\mu^2}{4(\lambda+\mu)(\lambda+2\mu)} \in \left(-\frac{5}{4};\ -\frac{1}{4}\right) \tag{3.9}$$

We emphasize that the asymptotic formulas (2.9) and (3.9) obtained concide with the zone $\alpha \sim \pi$ on the graph of the numerical solutions (see /5/, pp.962 and /1/, p.322).

In particular, there results from the formulas presented that under non-axisymmetric loading the index of the stress singularity can have a higher order than under axisymmetric loading.

Because of the appearance of an additional large parameter, all the representations found for the index of the stress singularity lose the asymptotic nature in two cases $\lambda \to +\infty$

or $\mu^0 \to +\infty$, which corresponds to an incompressible material of the matrix or an absolutely rigid inclusion. Both limit situations allow investigation within the framework of the asymptotic scheme applied in this paper and in /8/, but require separate examination.

4. A conical recess in a half-space. Let k_ε be a circular cone $\{\mathbf{x} : \theta < \arcsin \varepsilon\}$. Let us use the notation: $\mathbf{R}_1^3$ is the half-space $\{\mathbf{x} : x_3 < 1\}$, $\Omega_\varepsilon = \mathbf{R}_1^3 \setminus \bar{k}_\varepsilon$ is the half-space with the conical recess. We examine the problem of the deformation of a body Ω_ε subjected to axisymmetric normal loads p and q applied to the surface $\partial\Omega_\varepsilon$ near the recess edge (Fig.1). We introduce the coordinate $\mathbf{y} = (y_1, y_2, y_3)$ in the neighbourhood of the point **N**, where $y_j = \varepsilon^{-1}z^{-1}x_j$, $j = 1, 2$; $y_3 = \varepsilon^{-1}(z - 1)$. We assume that a force $q(\varepsilon, \mathbf{x}) = \varepsilon^{-2}q_0(r_y)$ acts on the surface $\{\mathbf{x}: z = 1\}$ while a load with intensity $p(\varepsilon, \mathbf{x}) = \varepsilon^{-3}p_0(y_3)$ acts on $\partial k_\varepsilon \cap \mathbf{R}_1^3$. Here $r_y = (y_1^2 + y_2^2)^{1/2}$ and q_0, p_0 are finite functions (the case of a concentrated load when p_0 or q_0 are proportional to the Dirac δ-function is not excluded). After changing to $\varepsilon = 0$ in the coordinates $\mathbf{y}$, the neighbourhood of the zone of force action is transformed into a half-space with a cutout cylinder $\mathbf{C} = D_1^2 \times \mathbf{R}^1$, where $D_1^2 = \{(y_1, y_2) : r_y < 1\}$ is a unit circle.

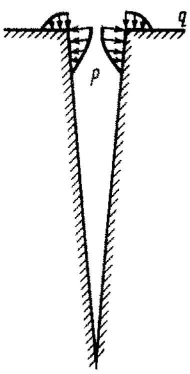

Fig.1

We assume there are no mass forces, i.e., the displacement vector $\mathbf{u}$ satisfies the homogeneous Lamé system. By virtue of axial symmetry, the problems $\mathbf{u}_\varphi = 0$ and $\sigma_{\varphi r}(\mathbf{u}) = \sigma_{\varphi z}(\mathbf{u}) = 0$ ((r, φ, z) are cylindrical coordinates $\varphi \in [0, 2\pi)$). The boundary conditions on the surface ∂k_ε and on the boundary of the half-space have the form

$$\sigma_{\theta\theta}(\mathbf{u}; \mathbf{x}) = -p(\varepsilon, \mathbf{x}), \quad \sigma_{\rho\theta}(\mathbf{u}; \mathbf{x}) = \sigma_{\theta\varphi}(\mathbf{u}; \mathbf{x}) = 0, \quad \mathbf{x} \in \partial k_\varepsilon \cap \mathbf{R}_1^3 \tag{4.1}$$

$$\sigma_{zz}(\mathbf{u}; \mathbf{x}) = -q(\varepsilon, \mathbf{x}), \quad \sigma_{zr}(\mathbf{u}; \mathbf{x}) = -\sigma_{z\varphi}(\mathbf{u}; \mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathbf{R}_1^3 \cap \Omega_\varepsilon \tag{4.2}$$

The approximate solution of the problem in found in Sects.5 and 6 for small ε, where different asymptotic methods are used; we will clarify the course of the discussion. The problem for an elastic half-space is the limit problem describing the state of stress and strain far from the recess. By virtue of the smallness of the zone of application of load p and q, they are here replaced by a concentrated effect. The analysis performed in Sect.5 for the three-dimensional boundary layer that occurs near the zone mentioned shows that in addition to the concentrated force determined according to the Saint-Venant principle, the singular solutions of higher order (the derivatives of Somigliani tensor columns) must be taken into account. According to Sect.2.2 /11/ and Chapter 4 /12/, a two-dimensional boundary layer that is found in the solution of the plane deformation problem occurs near a conical surface.

5. The limit problem in a half-space with a cylindrical cavity. As already mentioned, the domain Ω_ε is transformed into the set $\mathbf{R}_0^3 \setminus \mathbf{C}$ on changing to coordinates $\mathbf{y}$ near the edge of a conical recess. Let L be the operator of the Lamé system, and B and Γ the operators of the boundary coditions (4.1) and (4.2). In the coordinates $\mathbf{y}$ these operators are split into formal series in powers of ε. The formulas

$$L(\partial/\partial\mathbf{x})\Psi(\mathbf{y}) = \varepsilon^{-2}L(\partial/\partial\mathbf{y})\Psi(\mathbf{y}) + \varepsilon^{-1}L_1(\mathbf{y}, \partial/\partial\mathbf{y})\Psi(\mathbf{y}) + \ldots \tag{5.1}$$

$$B(\partial/\partial\mathbf{x})\Psi(\mathbf{y}) = \varepsilon^{-1}B_0(\mathbf{y}, \partial/\partial\mathbf{y})\Psi(\mathbf{y}) + B_1(\mathbf{y}, \partial/\partial\mathbf{y})\Psi(\mathbf{y}) + \ldots$$

$$\Gamma(\partial/\partial\mathbf{x})\Psi(\mathbf{y}) = \varepsilon^{-1}\Gamma(\partial/\partial\mathbf{y})\Psi(\mathbf{y}) + \Gamma_1(\mathbf{y}, \partial/\partial\mathbf{y})\Psi(\mathbf{y}) + \ldots$$

$$L_1\left(\mathbf{y}, \frac{\partial}{\partial\mathbf{y}}\right) = -2y_3L\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, 0\right) - \left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)L'\left(\frac{\partial}{\partial\mathbf{y}}\right) -$$

$$\left(y_3\frac{\partial}{\partial y_3} + 1\right)L'\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, 0\right); \quad B_0\left(\mathbf{y}, \frac{\partial}{\partial\mathbf{y}}\right) = \cos\varphi\sigma^{(1)} + \sin\varphi\sigma^{(2)}$$

$$B_1\left(\mathbf{y}, \frac{\partial}{\partial\mathbf{y}}\right) = -y_3B_0\left(\mathbf{y}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, 0\right) - \left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)B_0'\left(\mathbf{y}, \frac{\partial}{\partial\mathbf{y}}\right) - \sigma^{(3)};$$

$$\Gamma\left(\frac{\partial}{\partial\mathbf{y}}\right) = \sigma^{(3)}, \quad \Gamma_1\left(\mathbf{y}, \frac{\partial}{\partial\mathbf{y}}\right) = -y_3\Gamma\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, 0\right) -$$

$$\Gamma\left(0, 0, y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right); \quad \sigma^{(j)}(\Psi; \mathbf{y}) = \|\sigma_{jk}(\Psi; \mathbf{y})\|_{k=1}^3$$

are needed later.

The prime here denotes a derivative of the abstract function $\partial/\partial y_3 \to L(\partial/\partial\mathbf{y})$.

The solution of the initial problem near the point **N** is sought as the bounary layer $\varepsilon^{-2}\mathbf{W}^\circ(\mathbf{y}) + \varepsilon^{-1}\mathbf{W}^1(\mathbf{y})$. According to (5.1) and (4.1), (4.2), the vector-function $\mathbf{W}^\circ$ is subject to the equations

$$L(\partial/\partial\mathbf{y})\mathbf{W}^\circ(\mathbf{y}) = 0, \quad \mathbf{y} \in \mathbf{R}_0^3 \setminus \mathbf{C}, \quad \Gamma(\partial/\partial\mathbf{y})\mathbf{W}^\circ(\mathbf{y}) = 0, \quad \mathbf{y} \in \partial\mathbf{R}_0^3 \setminus \mathbf{C} \tag{5.2}$$

$$B_0(\mathbf{y}, \partial/\partial\mathbf{y})\mathbf{W}^\circ(\mathbf{y}) = -p_0(y_3)(\cos\varphi, \sin\varphi, 0), \quad \mathbf{y} \in \partial\mathbf{C} \cap \mathbf{R}_0^3$$

Similar programs were investigated in /13, 14/. Here only axisymmetric solutions occur; moreover, the asymptotic form as $|y| \to \infty$ is used later to accuracy $o(|y|^{-2})$. In the main the solution W° is represented in the form $W^\circ(y) = c_3 T^{(3)}(y) + o(|y|^{-1})$ where $T^{(j)}$ are solutions of problems on the action of a concentrated force in the direction $e^{(j)}$ on an elastic half-space (see /15/, p.237). However, the external forces from (5.2) are self-equilibrated, meaning $c_3 = 0$. The next of the asymptotic form of the axisymmetric solution has the form

$$W^\circ(y) = c_1(T^{(1)}_{,1}(y) + T^{(2)}_{,2}(y)) + O(|y|^{-3}\ln|y|), \quad |y| \to \infty \tag{5.3}$$

In order to find the dependence of the constant c_1 in (5.3) on the load p we will use the method of /10/. We first construct special solutions of the homogeneous problem (5.2) that have growth at infinity. The residual of the vector $V(y) = 2^{-1/2}(y_1, y_2, -2\lambda(\lambda + 2\mu)^{-1} y_3)$ in the homogeneous boundary condition on $\partial C \cap R_0^3$ is $\alpha(\cos\varphi, \sin\varphi, 0)$, where $\alpha = 2^{1/2}\mu(3\lambda + 2\mu)(\lambda + 2\mu)^{-1}$. This error is compensated by the axisymmetric solution αY of the elasticity theory problem for a plane with a cutout unit circle

$$Y_k(y) = (2\mu)^{-1} r_y^{-2} y_k, \; k = 1, 2; \; Y_3(y) = 0 \tag{5.4}$$

$$\sigma_{\varphi\varphi}(Y; y) = -\sigma_{rr}(Y; y) = r_y^{-2}$$

$$\sigma_{zz}(Y) = \sigma_{r\varphi}(Y) = \sigma_{rz}(Y) = \sigma_{\varphi z}(Y) = 0 \tag{5.5}$$

Since $\Gamma(\partial/\partial y) Y = 0$ on $\partial R_0^3 \setminus C$ by virtue of (5.5), the vector $\zeta = V + \alpha Y$ satisfies the homogeneous problem (5.2).

Proposition 5. The constant c_1 from the asymptotic form (5.3) is calculated from the formula

$$c_1 = -4\pi(\lambda + \mu)(\lambda + 2\mu)^{-1} P, \quad P = \int_{-\infty}^{0} p_0(t)\,dt \tag{5.6}$$

Proof. Let D_R^3 be a sphere of radius R with centre at the point $y = 0$. We substitute the fields W° and ζ into the Betti formula for the domain $(D_R^3 \cap R_0^3) \setminus C$. Taking account of the boundary conditions on ∂R_0^3 we have

$$\int_{S_1} \zeta\cdot\sigma^{(n)}(W^\circ) - W^\circ\cdot\sigma^{(n)}(\zeta)\,ds = \int_{S_2} \zeta\cdot\sigma^{(n)}(W^\circ) - W^\circ\cdot\sigma^{(n)}(\zeta)\,ds \tag{5.7}$$

$$\sigma^{(n)} = \sigma n, \; S_1 = (\partial C \cap R_0^3) \cap D_R^3, \; S_2 = (\partial D_R^3 \cap R_0^3) \setminus C$$

where n is the external normal. Taking account of the boundary conditions on $\partial C \cap R_0^3$ for the vector functions W°, ζ we find that to the left in (5.7) the integral can be extended to $\partial C \cap R_0^3$. According to (5.3), the right-hand side of (5.7) equals

$$c_1 \int_{\partial D_R^3 \cap R_0^3} \{V(y)\cdot\sigma^{(n)}(T^{(1)}_{,1} + T^{(2)}_{,2}; y) - (T^{(1)}_{,1}(y) + T^{(2)}_{,2}(y))\cdot\sigma^{(n)}(V; y)\}\,ds_y =$$

$$-c_1 \int_{D_R^3 \cap R_0^3} 2^{1/2} V(y)\cdot V(\partial/\partial y_1, \partial/\partial y_2, 0)\,\delta(y_1, y_2, 0)\,dy_1 dy_2 = 2^{1/2} c_1$$

with error $o(1)$ as $R \to \infty$.

Passing to the limit as $R \to \infty$, and evaluating the integral over $\partial C \cap R_0^3$ we obtain (5.6).

Let us construct the second term of a solution of boundary-layer type. We find by using (5.1) that the vector W^1 is determined from the problem

$$L(\partial/\partial y) W^1(y) = -L_1(y, \partial/\partial y) W^\circ(y), \; y \in R_0^3 \setminus C \tag{5.8}$$

$$B_0(y, \partial/\partial y) W^1(y) = -B_1(y, \partial/\partial y) W^\circ(y), \; y \in \partial C \cap R_0^3 \tag{5.9}$$

$$\Gamma(\partial/\partial y) W^1(y) = -q_0(r_y) e^{(3)} - \Gamma_1(y, \partial/\partial y) W^\circ(y), \; y \in \partial R_0^3 \setminus C \tag{5.10}$$

By virtue of (5.3) the right-hand sides of (5.8) and (5.10) are of the order $|y|^{-3}$ and $|y|^{-2}$, respectively, as $|y| \to \infty$. Consequently, according to /9, 10, 14/

$$W^1(y) = c_3 T^{(3)}(y) + \Upsilon(y) + O(|y|^{-2}\ln|y|), \quad |y| \to \infty \tag{5.11}$$

Here c_3 is a certain constant, Υ is a particular solution of the problem $L\Upsilon = -L_1\Xi$

in $\mathbf{R}_0^3$; $\Gamma\Upsilon = -\Gamma_1\Xi$ on $\partial\mathbf{R}_0^3 \setminus 0$; and Ξ denotes the expression $c_1(T_{,1}^{(1)} + T_{,2}^{(2)})$ from (5.3).

Proposition 6. The factor c_3 in the asymptotic form (5.11) is evaluated from the formula

$$c_3 = -2\pi Q - 2\mu(\lambda+\mu)^{-1}c_1, \quad Q = \int_1^{+\infty} q_0(t)\,dt \tag{5.12}$$

Formula (5.12) is proved by using the method of /10/; the same calculations are used as in Proposition 5 as well as the later representation of the vector $\Upsilon(\mathbf{y})$:

Proposition 7. The vector function Υ is determined by the equality $\Upsilon(\mathbf{y}) = y_3\{y_1\partial/\partial y_1 + y_2\partial/\partial y_2\}\,\Xi(\mathbf{y})$, and its components are homogeneous functions of degree -1.

6. ***The asymptotic form of the state of stress and strain in*** Ω_ε. Using the asymptotic expansions (5.3) and (5.11) and returning to the coordinates $\mathbf{x}$ and taking account of Proposition 6, we find that for $|\mathbf{x}-\mathbf{N}| = O(\varepsilon^{1/2})$ the following relationship holds:

$$\varepsilon^{-2}\mathbf{W}^\circ(\mathbf{y}) + \varepsilon^{-1}\mathbf{W}^1(\mathbf{y}) \sim c_3T^{(3)}(\mathbf{x}-\mathbf{N}) + c_1(T_{,1}^{(1)}(\mathbf{x}-\mathbf{N}) + T_{,2}^{(2)}(\mathbf{x}-\mathbf{N})) \tag{6.1}$$

Merging the three-dimensional boundary layer with the displacement field $\mathbf{v}$ that approximates the solution $\mathbf{u}$ far from k_ε, we conclude that $\mathbf{v}$ is a solution of the boundary-value problem

$$L\mathbf{v} = 0 \quad \text{in} \quad \mathbf{R}_1^3, \quad \Gamma\mathbf{v} = c_3\mathbf{e}^{(3)}\delta + c_1(\mathbf{e}^{(1)}\delta_{,1} + \mathbf{e}^{(2)}\delta_{,2}) \quad \text{on} \quad \partial\mathbf{R}_1^3$$

where $\mathbf{e}^{(j)}$ are unit vectors in $\mathbf{R}^3$ while the δ-function is concentrated at the point $\mathbf{x} = \mathbf{N}$. Therefore, $\mathbf{v}(\mathbf{x})$ agrees with the right-hand side of the relationship (6.1).

Thus, asymptotic representations of the solution have been found in the following two zones: in the immediate proximity of the section of the boundary where the external load acts and far from the cone k_ε. We will now construct additional terms that take account of the boundary conditions (4.1) and (4.2) outside the neighbourhood of the point $\mathbf{N}$ and the presence of the boundary singularity at the cone apex.

The vector $\mathbf{v}$ leaves the residual

$$\sigma_{\theta\theta}(\mathbf{v};\rho) = X(\rho) + O(\varepsilon), \quad \sigma_{\rho\theta}(\mathbf{v}) = \sigma_{\theta\varphi}(\mathbf{v}) = 0 \tag{6.2}$$
$$X(\rho) = \frac{\mu}{\lambda+\mu}\left(\frac{Q}{2(1-\rho)^3} - \frac{2}{\lambda+2\mu}P\left(\frac{\lambda+\mu}{(1-\rho)^3} + \frac{\mu}{(1-\rho)^3}\right)\right)$$

in the homogeneous boundary conditions (4.1) on $\partial k_\varepsilon \cap \mathbf{R}_1^3$

In order to eliminate the error (6.2) we construct the boundary layer $\varepsilon z\mathbf{w}(y_1, y_2, z)$. We emphasize that the quantities (6.2) are characterized by a "slow" dependence on z far from the point $\mathbf{N}$ and therefore, a two-dimensional boundary layer occurs (the extended variable $y_3 = \varepsilon^{-1}(x_3 - 1)$ was used in Sect.5 and the boundary layer was three-dimensional). As in Sect.2 we obtain that the components of $\mathbf{w}$ are solutions of problem on plane and antiplane deformation of the domain $\mathbf{R}^2 \setminus D_1^2$. Changing to coordinates (y_1, y_2, z) in (6.2), we have $\sigma_{\theta\theta}(\mathbf{v}; z) = X(z) + O(\varepsilon)$. Consequently, the boundary conditions on ∂D_1^2 for the two-dimensional vector (w_1, w_2) have the form

$$\sigma_{rr} = -X(z), \quad \sigma_{r\varphi} = 0 \tag{6.3}$$

This means that $\mathbf{w} = X(z)\,\mathbf{Y}(\mathbf{y})$, where $\mathbf{Y}$ is the vector of the function (5.4).

According to /9, 10/, the axisymmetric displacement field $\mathbf{u}$ allows of the expansion

$$\mathbf{u}(\varepsilon,\mathbf{x}) = c^{(0)}(\varepsilon)\,\mathbf{e}^{(3)} + c^{(1)}(\varepsilon)\,\rho^{\Lambda^{(1)}(\varepsilon)}\boldsymbol{\Phi}^{(1)}(\varepsilon,\theta,\varphi) + c^{(2)}(\varepsilon)\,\rho^{\Lambda^{(2)}(\varepsilon)}\boldsymbol{\Phi}^{(2)}(\varepsilon,\theta,\varphi) + \ldots \tag{6.4}$$

in the neighbourhood of the apex of the cone k_ε

Here $c^{(i)}(\varepsilon)$ are certain constants. The asymptotic form of the indices $\Lambda^{(i)}(\varepsilon)$ as $\varepsilon \to 0$ is determined by (2.9) and (3.9) while the angular parts $\boldsymbol{\Phi}^{(i)}$ have the form

$$\rho\Phi^{(i)}(0,\theta,\varphi) = b_1^{(i)}(x_1\mathbf{e}^{(1)} + x_2\mathbf{e}^{(2)}) + b_2^{(i)}x_3\mathbf{e}^{(3)} \tag{6.5}$$
$$b_1^{(1)} = b_2^{(2)} = 1, \quad b_1^{(2)} = 0, \quad b_2^{(1)} = -(5\lambda^2 + 9\mu\lambda + 2\mu^2)\,[4(\lambda+\mu)(\lambda+2\mu)]^{-1}$$

According to (6.4), (2.9) and (3.9), with the asymptotic representation $\mathbf{u}(\varepsilon,\mathbf{x}) \sim \mathbf{v}(\mathbf{x}) + \varepsilon z\mathbf{w}(y_1, y_2, z)$ found earlier, we conclude that in (6.4)

$$c^{(0)}(\varepsilon) = (2\lambda+3\mu)(4\pi\mu(\lambda+\mu))^{-1}(c_3 - c_1) + O(\varepsilon) \tag{6.6}$$
$$c^{(i)}(\varepsilon) = \pm(v_{1,1}(0)\,b_2^{(j)} - v_{3,3}(0)\,b_1^{(j)}) + O(\varepsilon), \quad i \neq j, \; i, j = 1, 2$$
$$v_{1,1}(0) = (2\lambda+\mu)(8\pi\mu(\lambda+\mu))^{-1}(2c_1 - c_3), \quad v_{3,3}(0) = (2\pi\mu)^{-1}(c_1 - c_3)$$

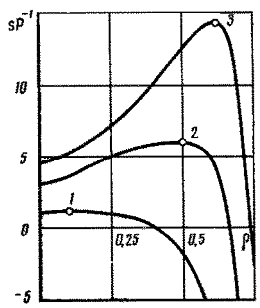

Fig.2

1°. According to /7, 1/ and Sect.3 of this paper, the index of stress singularity at the apex of a conical recess is $O(\varepsilon^2)$. According to the Novozhilov criterion /16/, such a stress singularity may be unimportant exert no influence on the nature of the fracture. In fact, the condition $(\text{mes}\,K)^{-1}\int_K \sigma_{ss}(\mathbf{x})ds > \sigma_c$ (for a conical surface $K = \{\rho < d, \theta = \theta_0\}$) means that

$$\sigma_0 < 2\Sigma(\theta_0)(\sin\theta_0)^{-1}(2 + \varepsilon^2\Lambda_2^{(2)} + O(\varepsilon^3))^{-1}\exp(\varepsilon^2|\ln d|(\Lambda_2^{(2)} + O(\varepsilon))) = \Sigma(\theta_0)(\sin\theta_0)^{-1} + O(\varepsilon^2|\ln d|) \tag{6.7}$$

Here $\Sigma(\theta_0)$ is a certain quantity evaluated by the formulas (6.4), (6.5), and (6.6), and d is a structural parameter /16/ of the material referred to the distance to the point of load application. If the remainder in (6.7) is small compared with the first term, the presence of the singularity exerts no influence; if $\varepsilon^2|\ln d| \gg 1$ (because of the smallness of d), then the pressence of the singularity is decisive.

2°. We will examine the part of the conical surface ∂k_ε between the apex O and the zone of load action (Fig.1). By virtue of (4.1) and (4.2) and the axial symmetry, only the stresses $\sigma_{\rho\rho}$ and $\sigma_{\varphi\varphi}$ differ from zero. They are found from (6.2), (6.3) and (5.5) and mainly (without taking account of the correction terms occurring in the immediate proximity of the apex O; Sect.1°) are

$$\sigma_{\rho\rho} = -6s(\rho);\ \sigma_{\varphi\varphi} = 2(1-2\nu)s(\rho),\ \nu = \lambda[2(\lambda+\mu)]^{-1} \tag{6.8}$$
$$s(\rho) = Q(2(1-\rho)^2)^{-1} - P(1-\nu)^{-1}((1-\rho)^{-3} + (1-2\nu)(1-\rho)^{-2})$$

Let the forces P and Q be directed within the body $(Q, P > 0)$. If $QP^{-1} < 4$ then the stresses $\sigma_{\rho\rho}$ are tensile and increase monotonically for $\rho \in (0, 1)$; the stresses $\sigma_{\varphi\varphi}$ are compressive. If $QP^{-1} \geqslant 4$, then the stresses $\sigma_{\rho\rho}$ are compressive in the neighbourhood of the apex, while $\sigma_{\varphi\varphi}$ are tensile. For $QP^{-1} > (5-4\nu)(1-\nu)^{-1} = \gamma_0$ there is a local maximum of $\sigma_{\varphi\varphi}$ at the point $\rho_0 = [1 - 3[(1-\nu)QP^{-1} - 2(1-2\nu)]^{-1}$ (see Fig.2, where a graph of the function $P^{-1}s$ is shown for $\nu = 1/3$ and the parameter QP^{-1} equal to 6, 10, 13 (curves *1*, *2*, *3*, respectively); the stresses $\sigma_{\varphi\varphi}$ and $\sigma_{\rho\rho}$ are evaluated from (6.8)). Therefore, taking account of the material in Sect.1° we conclude that fracture is possible at a distance from the apex O when $\varepsilon^2|\ln d| \ll 1$; it is characterized by the formation of fine surface cracks perpendicular to the circle $\{\rho = \rho_0, \theta = \arcsin\varepsilon\}$. As the ratio QP^{-1} increases from the value γ_0, the point ρ_0 moves away from the apex O to the boundary of the half-space.

We note that the effect of fracture zone shift from the cone apex was observed in experiments /17/ (see also /18/).

3°. The algorithm elucidated for the asymptotic solution of the problem of the deformation of a half-space with a conical recess also applies in the case of loading from inside the recess. (We emphasize that in this case the problem from Sect.5 is replaced by an analogous problem concerning a space with a cylindrical cavity; the computations are simplified here). Analysis of the appropriate formulas shows that in the case of such loading the stresses $\sigma_{\rho\rho}$ and $\sigma_{\varphi\varphi}$ decrease monotonically from the zone of application of the force p to the cone apex.

4°. The results of Sect.5 show that replacement of external loads distribution in a small zone by a concentrated force in an elastic half-space is not admissible: expression (6.1) containing derivatives of Somigliani tensor columns and the vector $\mathbf{e}_3 T^{(3)}$ corresponding to the problem of a concentrated force are quantities of the same order. However, all the coefficients of the linear combination are expressed in terms of the principal vectors P and Q of the external forces (formulas (5.6) and (5.12)).

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ON A CLASS OF EXACT SOLUTIONS OF A NON-AXISYMMETRIC CONTACT PROBLEM FOR AN INHOMOGENEOUS ELASTIC HALF-SPACE*

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A non-axisymmetric mixed boundary-value problem is considered concerning the pressure (in the absence of friction and adhesion forces) of a stiff circular-planform stamp with a base of aribitrary shape on an inhomogeneous elastic half-space. The shear modulus of the half-space material is constant while Poisson's ratio is an arbitrary piecewise-continuous function of the depth. By using the theory of dual integral equations associated with the generalized Hankel integral operator, the problem is reduced to a sequence of one-dimensional Fredholm integral equations of the second kind.

It is shown that the integral equations obtained allow exact solutions to be constructed for periodic law of variation of the half-space material elastic properties with depth. The solution of a non-axisymmetric problem regarding the eccentric impression of a stamp with a flat base is presented as a example, on the basis of which the influence of inhomogeneity of the elastic material on the magnitude of the stamp displacement parameters is investigated. An asymptotic analysis is performed for the solution in the case when the elastic characteristics of the material become rapidly oscillating functions.

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